EN5101 Digital Control Systems
Kalman Filter

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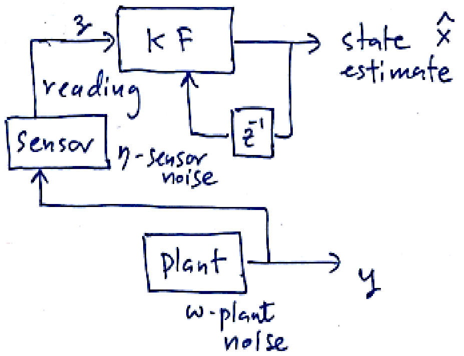
Introduction

- Kalman filter is an algorithm that uses a series of data observed over time, which contains noise and other inaccuracies, to estimate unknown variable accurately.
- It was proposed by R E Kalman in 1960, and became a standard approach for optimal estimation
- KF is a real-time, recursive, efficient estimation algorithm in standard (KF), extended (EKF), an unscenrted (UKF) forms

KF Highlights

- minimum variance
- zero mean error
- recursive



$\hat{x} \simeq y$

$E[\hat{x}-y] = 0$ zero bias (mean error)

$Cov[\hat{x}-y] \rightarrow$ minimize

Preliminaries: Definitions and Identities

z - scalar $\underline{x}$ - column vector then,

$$\frac{\partial z}{\partial \underline{x}} = \begin{bmatrix} \partial z/\partial x_1 \\ \partial z/\partial x_2 \\ \vdots \\ \partial z/\partial x_n \end{bmatrix}_{n\times 1} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Likewise for matrix $A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$

$$\frac{\partial z}{\partial A} = \begin{bmatrix} \partial z/\partial a_{11} & \partial z/\partial a_{12} & \cdots & \partial z/\partial a_{1n} \\ \vdots & & & \\ \partial z/\partial a_{m1} & \cdots & & \partial z/\partial a_{mn} \end{bmatrix}$$

$$\frac{\partial x^T y}{\partial x} = y = \frac{\partial y^T x}{\partial x}$$

say $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $y = \begin{bmatrix} y_1 \\ \vdots \\ y \end{bmatrix}$ then

$$x^T y = [x_1 \; x_2 \cdots x_n]\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

$$\frac{\partial x^T y}{\partial x_1} = y_1 \qquad \frac{\partial x^T y}{\partial x_2} = y_2 \qquad \vdots \qquad \frac{\partial x^T y}{\partial x_n} = y_n$$

$$\begin{bmatrix} \frac{\partial x^T y}{\partial x_1} \\ \frac{\partial x^T y}{\partial x_2} \\ \vdots \\ \frac{\partial x^T y}{\partial x_n} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \\ y_n \end{bmatrix}$$

$$\frac{\partial x^T y}{\partial x} = y$$

$$\frac{\partial X^T N X}{\partial X} = 2NX$$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $N = \begin{bmatrix} & \end{bmatrix}_{n \times n}$ symmetric

e.g. $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $N = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$X^T N X = [x_1 \; x_2]\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [(ax_1 + bx_2) \; (bx_1 + cx_2)]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1(ax_1 + bx_2) + x_2(bx_1 + cx_2)$$

$$= ax_1^2 + 2bx_1x_2 + cx_2^2$$

Then,

$$\frac{\partial X^T N X}{\partial x_1} = 2ax_1 + 2bx_2$$

$$\frac{\partial X^T N X}{\partial x_2} = 2bx_1 + 2cx_2$$

$$\frac{\partial X^T N X}{\partial X} = \begin{bmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{bmatrix} = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2NX$$

Kalman Filter- Priliminaries

Model

process : $X_{k+1} = \Phi_{k+1} X_k + w_k$ — (a)

measurement : $z_k = H_k X_k + v_k$ — (b)

Assumptions : $E[w_k] = E[v_k] = 0$ zero mean noise

$Cov[w, v] = 0$ uncorrelated noise

$cov[w, w] = Q$ process noise covariance

$cov[v, v] = R$ sensor noise covariance.

Kalman Filter Equations

1. state extrapolation

$$\hat{x}^-_{k+1} = \Phi_{k+1}\hat{x}_k$$

2. Covariance extrapolation

$$P^-_{k+1} = \Phi_{k+1} P_k \Phi^T_{k+1} + Q_{k+1}$$

Priori (predictor)

3. Kalman gain computation

$$K_{k+1} = P^{-}_{k+1} H^T_{k+1}\left[H_{k+1} P^-_{k+1} H^T_{k+1} + R_{k+1}\right]^{-1}$$

4. state update

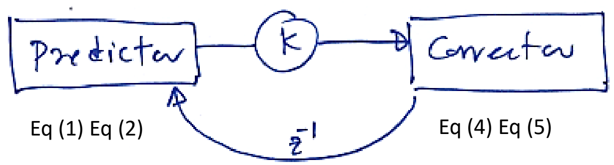
$$\hat{x}_{k+1} = \hat{x}^-_{k+1} + K\left[z_{k+1} - H_{k+1}\hat{x}^-_{k+1}\right]$$

5. covariance update

$$P_{k+1} = P^-_{k+1} - K_{k+1} H_{k+1} P^-_{k+1}$$

posteriori (corrector)

Kalman: Predictor Corrector



Kalman Filter Derivation

Step 1 : State Extrapolation $\quad \hat{x}^-_{k+1} = \Phi_{k+1}\hat{x}_k$

For $\hat{x}_{k+1}$ to be unbiased

$$E\left[\hat{x}_{k+1} - x_{k+1}\right] = 0$$

we assume that

$$\hat{x}_{k+1} = K'_{k+1}\hat{x}_k + K_{k+1} z_{k+1} \quad \text{— (c)}$$

previous estimate ($\hat{x}_k$), observation (z_{k+1})

the previous estimate and current measurement can be linearly combined to produce present best estimate

two weighting gains

$$E\Big[\underbrace{K'_{k+1}\hat{x}_k + K_{k+1} z_{k+1}}_{\hat{x}_{k+1}} - x_{k+1}\Big] = 0$$

(b) add & subtract

$$E\left[K'_{k+1}\hat{x}_k + K_{k+1}\left(H_{k+1}x_{k+1} + v_{k+1}\right) - x_{k+1} - K'_{k+1}x_k + K'_{k+1}x_k\right] = 0$$

(a) (a)

$$E\left[K'_{k+1}(\hat{x}_k - x_k) + K_{k+1}\left\{H_{k+1}\left(\Phi_{k+1}x_k + w_{k+1}\right) + v_{k+1}\right\} - \left(\Phi_{k+1}x_k + w_{k+1}\right) + K'_{k+1}x_k\right] = 0$$

$$E\left[K'_{k+1}(\hat{x}_k - x_k) + \left\{K_{k+1}H_{k+1}\Phi_{k+1} - \Phi_{k+1} + K'_{k+1}\right\}x_k + \left(K_{k+1}H_{k+1} - I\right)w_{k+1} + K_{k+1}v_k\right] = 0$$

$$K'_{k+1}E\left[\hat{x}_k - x_k\right] + (\quad)E\left[x_k\right] + (\quad)E\left[\cancel{w_{k+1}}\right]^{0} + K_{k+1}E\left[\cancel{v_k}\right]^{0} = 0$$

If this has to be zero this has to be zero as condition

Therefore for $E[\hat{X}_{k+1} - X_{k+1}] = 0$ the condition is

$$K_{k+1} H_{k+1} \Phi_{k+1} - \Phi_{k+1} + K'_{k+1} = 0$$

or

$$K'_{k+1} = (I - K_{k+1} H_{k+1}) \Phi_{k+1} \quad — (d)$$

This relationship of K and K' will make sure that the estimate is unbiased. However, we still don't have an explicit expression to calculate K, or K'.

Step 2 : Covariance Extrapolation

$$P_{k+1} = E[e_{k+1} \, e^T_{k+1}] \qquad e_{k+1} = \underset{\text{estimate}}{\hat{X}_{k+1}} - \underset{\text{actual value}}{X_{k+1}}$$

$$= \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_n^2 \end{bmatrix}$$

$$\text{Trace}(P_{k+1}) = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$$

$$\min[\text{Trace}(P_{k+1})] = \min[\sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2] \Rightarrow \text{Find } K_{k+1}.$$

Priori estimate error is

$$e^-_{k+1} = \hat{X}^-_{k+1} - X_{k+1}$$

$$\overset{(1)}{=} \Phi_{k+1} \hat{X}_k - \Phi_{k+1} X_k + w_{k+1}$$

$$= \Phi_{k+1} \underbrace{(\hat{X}_k - X_k)}_{\text{posterior error at } k \; e_k} + w_{k+1}$$

priori error at $k+1$

$$e^-_{k+1} = \Phi_{k+1} e_k + w_{k+1}$$

$$P^-_{k+1} = \Phi_{k+1} E[e_k e_k^T] \Phi^T_{k+1} + E[w_{k+1} w^T_{k+1}]$$

$$P^-_{k+1} = \Phi_{k+1} P_k \Phi^T_{k+1} + Q_{k+1} \quad — (2)$$

Step 4: Kalman Gain Calculation

$$\hat{X}_{k+1} = (I - K_{k+1} H_{k+1}) \underbrace{\Phi_{k+1} \hat{X}_k}_{\hat{X}^-_{k+1}} + K_{k+1} z_{k+1}$$

Then

$$e_{k+1} = (I - K_{k+1} H_{k+1}) \hat{X}^-_{k+1} + K_{k+1} z_{k+1} - X_{k+1}$$

$$= (I - K_{k+1} H_{k+1}) \hat{X}^-_{k+1} + K_{k+1} \overbrace{(H_{k+1} X_{k+1} + v_{k+1})}^{2(b)} - X_{k+1}$$

$$= \hat{X}^-_{k+1} - K_{k+1} H_{k+1} \hat{X}^-_{k+1} + K_{k+1} H_{k+1} X_{k+1} + K_{k+1} v_{k+1} - X_{k+1}$$

$$= (\hat{X}^-_{k+1} - X_{k+1}) - K_{k+1} H_{k+1} (\hat{X}^-_{k+1} - X_{k+1}) + K_{k+1} v_{k+1}$$

$$= (I - K_{k+1} H_{k+1})(\hat{X}^-_{k+1} - X_{k+1}) + K_{k+1} v_{k+1}$$

$$= (I - K_{k+1} H_{k+1}) e^-_{k+1} + K_{k+1} v_{k+1}$$

$$= (I - K_{k+1}H_{k+1})\, e^-_{k+1} + K_{k+1} v_{k+1}$$

$$E[e_{k+1}, e^T_{k+1}] = (I - K_{k+1}H_{k+1})\, E[e^-_{k+1}, e^{-T}_{k+1}]\, (I - K_{k+1}H_{k+1})^T + K_{k+1} E[v_{k+1}, v^T_{k+1}]\, K^T_{k+1}$$

$$P_{k+1} = (I - K_{k+1}H_{k+1})\, P^-_{k+1} \underset{\uparrow\ (I - K_{k+1}H_{k+1})^T}{} + K_{k+1} R_{k+1} K^T_{k+1} \qquad (9)$$

lets simplify notation by droping $k+1$ subscript

Then

$$P = (1 - KH)\, P^- (1 - KH)^T + KRK^T$$

$$= (1 - KH)\, P^- (1 - H^T K^T) + KRK^T$$

$$= P^- - KHP^- - P^- H^T K^T + KHP^- H^T K^T + KRK^T$$

$$= P^- - KHP^- - P^- H^T K^T + KHP^- H^T K^T + KRK^T$$

Take the trace

$$Tr(P) = Tr(P^-) - 2\, Tr(KHP^-) + Tr[K(HP^-H^T)K^T] + Tr(KRK^T)$$

and differentiate w.r.t. K

$$\frac{\partial Tr(P)}{\partial K} = \underbrace{\frac{\partial Tr(P^-)}{\partial K}}_{0} - 2\, \underbrace{\frac{\partial Tr(KHP^-)}{\partial K}}_{(HP^-)^T} + \underbrace{\frac{\partial Tr[K(HP^-H^T)K^T]}{\partial K}}_{2KHP^-H^T} + \underbrace{\frac{\partial Tr(KRK^T)}{\partial K}}_{2KR}$$

0 assuming minimum covariance in priori estimate

from identity $\frac{\partial Tr\, AC}{\partial A} = C^T$

from identity $\frac{\partial Tr\, ABA^T}{\partial A} = 2AB$

Therefore,

$$\frac{\partial Tr(P)}{\partial K} = -2 P^{-T} H^T + 2KHP^-H^T + 2KR = 0$$

when covariance is minimized, thus

$$-P^{-T}H^T + KHP^-H^T + KR = 0$$

$$K = P^{-T}H^T (HP^-H^T + R)^{-1} \quad — \quad (3)$$

This equation will calculate kalman gain so that the error estimate will be both zero mean and minimum covariance.

Step 4: State Update

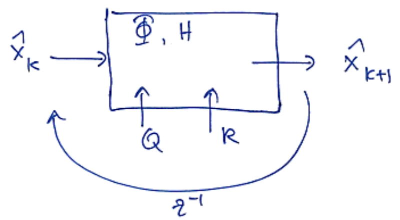
from (c) with substitution for K'_{k+1} from (d)

$$\hat{x}_{k+1} = (I - K_{k+1}H_{k+1}) \underbrace{\Phi_{k+1} \hat{x}_k}_{\hat{x}^-_{k+1}} + K_{k+1} z_{k+1}$$

Step 5: Covariance Update

put (3) onto (9) and obtain

$$P_{k+1} = (I - K_{k+1}H_{k+1})\, P^-_{k+1} \quad — \quad (5)$$



Assignment

(a) Describe the two basic properties of Kalman filter

(b) Write down the Kalman filter algorithm describing each of its steps.

(c) A spacecraft is accelerated by bursts of gas ejected by its reaction control thrusters. Position, velocity and acceleration of the spacecraft are denoted by $p(k)$, $\dot{p}(k)$, and $a(k)$ at discrete time k.

(i) Write expressions for $p(k+1)$, and $\dot{p}(k+1)$ and show that the state space model of the spacecraft is

$$\begin{bmatrix} p(k+1) \\ \dot{p}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p(k) \\ \dot{p}(k) \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} a(k)$$, where T is sampling interval.

(ii) An anomaly happens in the reaction control system of the spacecraft, which causes the gas thruster to eject random bursts of gas that results in zero mean random acceleration with standard deviation $\sigma_a = 2\text{m/s}^2$. Show that the process noise (random acceleration) covariance matrix of the system is

$$Q = \sigma_a^2 \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}.$$

(iii) Position of the spacecraft is measured by the position gyro as $z(k) = p(k) + v(k)$, where $v(k)$ is the zero mean gyro sensor noise with standard deviation $\sigma_v = 1\,\text{m}$. Before the anomaly happened the spacecraft was on stable orbit (at equilibrium point at $k = 0$). And, at the first two instants followed by the anomaly, i.e., $k = 1$ and $k = 2$, position was measured by the gyro sensor as $z_1 = 0.1\,\text{m}$, and $z_2 = -0.2\,\text{m}$. Use Kalman filter and estimate the position and velocity of the spacecraft at these two instants. The sampling interval of the spacecraft control system is $T = 1\,\text{s}$.

(iv) The first two random gas bursts caused $a(0) = 0.15\text{m/s}^2$ and $a(1) = -0.8\text{m/s}^2$ accelerations. Calculate the actual position of the spacecraft $p(1)$ and $p(2)$ at the first two sampling instants.

(v) Tabulate the actual position, measured position, and position estimated by Kalman filter. Comment on the results.